**Sequences**

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The sequence $(a_n)$ is defined by the formulas

$a_0 = \dfrac{1}{2}$ and $a_{n+1} = \dfrac{2a_n}{1+a_n^2}$ for $n > 0,$ and the sequence $(c_n)$ is defined by the formulas

$c_0 = 4$ and $c_{n+1} = c_n^2 - 2c_n + 2$ for $n \geq 0.$

Prove that $a_n = \dfrac{2c_0c_1\ldots c_{n-1}}{c_n}.$

**Solution by Arkady Alt , San Jose, California, USA.**

First note that $a_n < 1, n \geq 0.$ Indeed, $a_0 < 1$ and for any $n \geq 0$ assuming $a_n < 1$

we obtain $a_{n+1} = \dfrac{2a_n}{1+a_n^2} < 1 \Leftrightarrow (a_n - 1)^2 > 0.$ Also note that $1 - a_{n+1} =$

$1 - \dfrac{2a_n}{1+a_n^2} = \dfrac{(1-a_n)^2}{a_n^2+1}$ and $1 + a_{n+1} = 1 + \dfrac{2a_n}{1+a_n^2} = \dfrac{(1+a_n)^2}{a_n^2+1}.$

Hence, $\dfrac{1+a_{n+1}}{1-a_{n+1}} = \left(\dfrac{1+a_n}{1-a_n}\right)^2, n \geq 0$ and since $\dfrac{1+a_0}{1-a_0} = 3$ then $\dfrac{1+a_n}{1-a_n} = 3^{2^n}, n \geq 0.$

Indeed, for any $n \geq 0$ assuming $\dfrac{1+a_n}{1-a_n} = 3^{2^n}$ we obtain $\dfrac{1+a_{n+1}}{1-a_{n+1}} = (3^{2^n})^2 = 3^{2^{n+1}}.$

Thus, by Math Induction $\dfrac{1+a_n}{1-a_n} = 3^{2^n}$ for any $n \in \mathbb{N} \cup (0)$ and, therefore, $a_n = \dfrac{3^{2^n}-1}{3^{2^n}+1}.$

Also, since $c_{n+1} = c_n^2 - 2c_n + 2 \Leftrightarrow c_{n+1} - 1 = (c_n - 1)^n$ and $c_0 = 4 - 1 = 3$ then again

by Math Induction $c_n - 1 = 3^{2^n} \Leftrightarrow c_n = 3^{2^n} + 1 = \dfrac{3^{2^{n+1}}-1}{3^{2^n}-1}, n \in \mathbb{N} \cup (0)$ and, therefore,

$$\prod_{k=0}^{n-1} c_k = \prod_{k=0}^{n-1} \frac{3^{2^{k+1}}-1}{3^{2^k}-1} = \frac{3^{2^n}-1}{3^{2^0}-1} = \frac{3^{2^n}-1}{2} = \frac{a_n(3^{2^n}+1)}{2} = \frac{a_nc_n}{2} \Leftrightarrow a_n = \frac{2c_0c_1\ldots c_{n-1}}{c_n}$$